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The wave force on an infinitely long circular cylinder in an oblique sea

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An infinitely long circular cylinder is fixed with its generators horizontal so that it is half-immersed, with its axis lying in the free surface of water. A regular train of water waves is incident on the cylinder from an arbitrary horizontal direction, and is partly reflected and partly transmitted under the cylinder. In the present paper we are concerned with the vertical component of the wave acting on the cylinder. It is assumed that the fluid is inviscid, that the fluid motion is irrotational, and that the depth of water is infinite. The equations of motion are linearized, and surface tension is neglected.

We shall find it convenient to use the fact that the required vertical force component can be inferred from the solution of a related problem, which we shall call the *generalized heaving problem*. In this latter problem a certain normal velocity is prescribed on the cylinder so that water waves which travel obliquely outwards are generated. There are no waves incident from infinity. When the prescribed velocity has the same phase everywhere on the cylinder the waves travel normally outwards, and in this case the generalized heaving problem reduces to the ordinary heaving problem, on which much information is already available. The generalized problem is solved here by a method which is a generalization of the known method (Ursell 1949) for ordinary heaving (when the wave crests are parallel to the cylinder axis). Generalized-added-mass coefficients and generalized-wave-making parameters for generalized heaving are computed for a range of wavenumbers and angles of travel, and are extended to larger wavenumbers by means of asymptotic analysis. Reciprocity relations (the Haskind relations) are then used to obtain the vertical force component in the original transmission problem from the wave-making parameters of the generalized heaving problem.

1. Introduction

Although the translational motion of a ship is nonlinear, the oscillatory motion of the ship due to incident waves will be expected to be linear when the incident waves are small enough. Experiments have shown that this is actually the case. (References for the statements in this introduction are given in Ursell (1968).)

It is obvious that the response coefficients must involve properties of the translational motion and cannot therefore be calculated rigorously in the present state of our knowledge. To correlate theory and experiment a semi-empirical approach (strip theory) is sometimes used: the ship is divided into numerous segments by vertical planes normal to the ship's axis (these segments are known as strips), and the hydrodynamic force on the immersed surface of each strip is approximated by the force on the corresponding strip of an infinitely long cylinder of the same cross-section and oscillating in the same manner. In this way the calculation of the virtual mass and damping of the ship is made to depend on the calculation of a class of two-dimensional linear boundary-value problems. The forward speed of the ship enters into the calculation only through the modification of period due to forward speed, i.e. the period of encounter of the ship in the waves, and possibly through the modification of boundary conditions due to forward speed. The results of such calculations can then be compared with experiments, and it is hoped that this method will throw light on the interaction of ships and waves. Since the equations of oscillatory motion are assumed to be linear the calculation can be divided into three stages.

(i) To find the wave force on a *fixed* cylinder due to incident waves.

(ii) To integrate the forces on each strip over the length of the ship, and thus to find the total exciting force on the ship.

(iii) To calculate the response of the ship to the total exciting force.

In the present paper we shall be concerned only with the first stage, of finding the force on a fixed horizontal cylinder due to incident waves.

The most important application is to ships in *head seas* (when the wave crests are normal to the axis of the ship). It has, however, been shown that, according to the linear theory, head seas cannot travel along an infinite cylinder without change of form (Ursell 1968), and strip theory is therefore not immediately applicable. The case of *beam seas* (when the wave crests are parallel to the axis of the ship) is the only one that has so far been studied, and it is the added mass and damping parameters appropriate to beam seas that have been used in strip theory even when the incident waves are not beam seas. (For a discussion see Newman (1970).) In the present paper we shall consider the vertical force per unit axial length on an infinitely long fixed horizontal cylinder when the waves are obliquely incident. (The horizontal force can be treated similarly but is not considered in the present paper.) The calculation is simplest for a half-immersed circular cross-section, and this is the problem with which we shall be concerned. The velocity potential can be expanded in an infinite series which is analogous to the series for beam seas but more complicated, and which involves two parameters (wavenumber and angle of incidence). For each pair of parameters the pressure at any point of the cylinder can be found in principle by solving infinite systems of linear equations in an infinite number of unknowns; in practice these systems have to be replaced by a large finite system of equations. The required vertical force component is a linear combination of the unknowns. We shall see in § 6 below that the same vertical force component can also be obtained from the solution of a different but related problem, the problem of *generalized heaving* described in the next section. The latter problem has the advantage that asymptotic results for short wavelengths are much more readily obtained, and that numerical results can be compared with the well-established special case of ordinary heaving studied by Ursell (1949) and many later writers.

2. Statement of the mathematical problem

Let the origin of a rectangular co-ordinate system be taken in the mean free surface of the fluid, and let the axes be chosen so that the x and z co-ordinates are horizontal and the y co-ordinate is vertical and increasing with depth. Let associated cylindrical polar co-ordinates (r, θ, z) be defined so that

$$x = r\sin\theta, \quad y = r\cos\theta, \quad z = z.$$

It is supposed that a half-immersed circular cylinder occupies the region $r \leqslant a$, $-\infty < z < \infty$. The fluid will be treated as inviscid; then the motion, if originally irrotational, remains irrotational, and a velocity potential $\phi(x, y, z, t)$ exists. The equation of continuity is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z,t) = 0 \quad \text{in} \quad r > a, \quad y > 0. \tag{2.1}$$

The linearized condition of constant pressure at the free surface is

$$\left(\frac{\partial^2}{\partial t^2}-g\frac{\partial}{\partial y}\right)\phi(x,y,z,t) = 0 \quad \text{when} \quad y = 0, \quad |x| > a;$$

see Lamb (1932, § 227). When the motion is simple harmonic, of constant period $2\pi/\omega$, this free-surface condition takes the form

$$K\phi + \partial\phi/\partial y = 0 \quad \text{when} \quad y = 0, \quad |x| > a, \tag{2.2}$$

where $K = \omega^2/g$. Additional boundary conditions are prescribed on the immersed surface ($r = a$, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, $-\infty < z < \infty$) of the circular cylinder, and also at infinity. Two problems will be considered.

(i) The scattering problem. The velocity potential is of the form

$$\Phi_f(x,y,z,t) = \phi_f(x,y)\exp(iKz\cos\mu - i\omega t) \tag{2.3}$$

$$= \phi_f(x,y)\exp(ikz - i\omega t), \quad \text{say}, \tag{2.4}$$

where $$k = K\cos\mu,$$
and where

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 - k^2)\,\phi_f(x,y) = 0 \quad \text{when} \quad r > a, \quad y > 0; \tag{2.5}$$

$$K\phi_f + \partial\phi_f/\partial y = 0 \quad \text{when} \quad y = 0, \quad |x| > a; \tag{2.6}$$

$$\partial\phi_f/\partial r = 0 \quad \text{when} \quad r = a, \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi; \tag{2.7}$$

$$\phi_f(x,y) \to \begin{cases} \dfrac{gb_1}{\omega}\exp(-Ky + iKx\sin\mu) + \dfrac{gb_1'}{\omega}\exp(-Ky - iKx\sin\mu) \\ \qquad\qquad\qquad \text{when} \quad x \to -\infty, & (2.8) \\ \dfrac{gb_1''}{\omega}\exp(-Ky + iKx\sin\mu) \quad \text{when} \quad x \to +\infty, & (2.9) \end{cases}$$

where b_1, b_1' and b_1'' are respectively the amplitudes of the incident, reflected and transmitted waves. It is seen that the angle μ describes the direction of propagation of these waves; for beam seas $\mu = \frac{1}{2}\pi$.

(ii) The generalized heaving problem. The potential is of the form

$$\Phi(x,y,z,t) = \phi(x,y)\exp(ikz-i\omega t), \tag{2.10}$$

where

$$(\partial^2/\partial x^2+\partial^2/\partial y^2-k^2)\,\phi(x,y) = 0 \quad \text{when} \quad r > a,\ y > 0; \tag{2.11}$$

$$K\phi+\partial\phi/\partial y = 0 \quad \text{when} \quad y = 0,\ |x| > a; \tag{2.12}$$

$$\partial\phi/\partial r = -(\omega b/Ka\pi)(A+iB)\cos\theta \quad \text{when} \quad r = a,\ -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi; \tag{2.13}$$

$$\phi(x,y) \to (gb/\omega)\operatorname{cosec}\mu\exp(-Ky+iK|x|\sin\mu) \quad \text{when} \quad |x| \to \infty, \tag{2.14}$$

where $b\operatorname{cosec}\mu$ is the wave amplitude at infinity. It is convenient to treat b as prescribed, and $A(Ka,\mu)$ and $B(Ka,\mu)$ as parameters which are to be determined. The boundary condition (2.13) describes a flexural wave which travels along the surface of the cylinder and generates an oblique wave in the water. The generalized heaving problem has no immediate physical application, but it is somewhat easier to compute than the scattering problem, and it can be readily compared with the well-known special case of beam seas. Also, the Haskind relation (§ 6 below) shows that the vertical force on the fixed cylinder in the scattering problem can be deduced from the parameters A and B of the generalized heaving problem. The present paper (except for § 6 below) will accordingly be mainly concerned with the generalized heaving problem.

3. Expansion of the heaving potential $\phi(x,y)$

It is convenient to introduce a positive quantity β which satisfies the equations

$$\cosh\beta = K/k = \sec\mu, \quad \sinh\beta = \tan\mu.$$

The potential will be expanded as the sum of a suitable wave-source potential ϕ_0 and of suitable wave-free potentials ϕ_m, derived (in a different notation) in an earlier paper (Ursell 1968). We write

$$\phi_0(kr,\theta) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cosh t}{\cosh t-\cosh\beta}\exp\{-kr\cosh(t-i\theta)\}\,dt, \tag{3.1}$$

where the contour of integration passes above the pole $t = -\beta < 0$ and below the pole $t = \beta > 0$. We have the expansion (Ursell 1962, equation (2.13))

$$\begin{aligned}\phi_0(kr,\theta) = {} & (\pi i-\beta)\coth\beta\left\{I_0(kr)+2\sum_{m=1}^{\infty}(-1)^m\cosh m\beta\, I_m(kr)\cos m\theta\right\}\\ & +K_0(kr)-2\coth\beta\sum_{m=1}^{\infty}(-1)^m\sinh m\beta\left[\frac{\partial}{\partial\nu}(I_\nu(kr)\cos\nu\theta)\right]_{\nu=m},\end{aligned} \tag{3.2}$$

where the functions $I_\nu(kr)$ and $K_0(kr)$ are the usual Bessel functions of imaginary argument (Watson 1944, chap. 3). Their series expansions are quoted by Ursell (1962, p. 502). Also (Ursell 1968, p. 814)

$$\phi_0(kr,\theta) \to \pi i\operatorname{cosec}\mu\exp(-Ky+iK|x|\sin\mu) \quad \text{when} \quad |x| \to \infty, \tag{3.3}$$

and the radiation condition at infinity is satisfied. We also write

$$\phi_m(kr,\theta) = K_{2m}(kr)\cos 2m\theta + (2/\cos\mu)\, K_{2m-1}(kr)\cos(2m-1)\,\theta + K_{2m-2}(kr)\cos(2m-2)\,\theta \quad (m = 1, 2, 3, \ldots). \tag{3.4}$$

These functions are exponentially small at infinity. The functions ϕ_0 and ϕ_m satisfy (2.11) and (2.12). We now write

$$\phi = \frac{\omega b}{K\pi}\left[-i\phi_0 + 2\sum_{m=1}^{\infty}\{p_{2m}(Ka,\mu) + iq_{2m}(Ka,\mu)\}\frac{(\frac{1}{2}ka)^{2m}}{(2m-1)!}\phi_m\right],$$

where the unknown real-valued coefficients p_{2m}, q_{2m}, A and B are to be determined from the boundary condition (2.13), which reduces to

$$-ia\left(\frac{\partial\phi_0}{\partial r}\right)_{r=a} + 2a\sum_{m=1}^{\infty}\{p_{2m} + iq_{2m}\}\frac{(\frac{1}{2}ka)^{2m}}{(2m-1)!}\left(\frac{\partial\phi_m}{\partial r}\right)_{r=a} = -(A+iB)\cos\theta, \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{3.5}$$

On account of (3.3) the radiation condition (2.14) at infinity is automatically satisfied. Let the operators

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos 2m\theta \ldots d\theta \quad (m = 0, 1, 2, \ldots)$$

be applied to this equation. In this way two infinite systems of simultaneous linear equations are obtained, one system involving the unknowns A, p_2, p_4, ..., the other the unknowns B, q_2, q_4, ... For any given value of Ka and any given value of μ these systems are solved approximately by retaining only a finite number of unknowns and the same number of equations.

The vertical upward force per unit length of the cylinder is

$$i\rho a\omega \exp(ikz - i\omega t)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\phi(a\sin\theta, a\cos\theta)\cos\theta\, d\theta, \tag{3.6}$$

which is equivalent to

$$-2\rho a\frac{b\omega^2}{K\pi}(L - iM)\exp(ikz - i\omega t), \quad \text{say,} \tag{3.7}$$

where
$$L(Ka,\mu) = \frac{K\pi}{2b\omega}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\operatorname{Im}\phi)\cos\theta\, d\theta \tag{3.8}$$

and
$$M(Ka,\mu) = \frac{K\pi}{2b\omega}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\operatorname{Re}\phi)\cos\theta\, d\theta \tag{3.9}$$

are non-dimensional parameters. The force (3.6) can be resolved into components respectively in phase and in quadrature with the heaving velocity, i.e. with $A + iB$. More precisely, we write

$$L - iM = (A+iB)(C_v - iC_a), \tag{3.10}$$

whence
$$C_a = \frac{MA+LB}{A^2+B^2}, \quad C_v = \frac{LA-MB}{A^2+B^2}. \tag{3.11}$$

C_a is evidently an added-mass coefficient, and C_v is a damping coefficient which can be related to the *amplitude ratio*, defined to be

$$\frac{\text{Wave amplitude at infinity}}{\text{Amplitude of oscillation of the cylinder}} = \frac{b \operatorname{cosec} \mu}{(b/Ka\pi)\,|A+iB|} = \frac{\pi Ka \operatorname{cosec} \mu}{(A^2+B^2)^{\frac{1}{2}}}. \tag{3.12}$$

In fact, on equating the work done by the cylinder to the energy carried to infinity by the waves it is found that

$$LA - MB = (A^2+B^2)\,C_v = \tfrac{1}{2}\pi^2 \operatorname{cosec} \mu, \tag{3.13}$$

and it follows that

$$C_v = \frac{1}{2}\frac{\sin \mu}{(Ka)^2}\left(\frac{Ka\pi \operatorname{cosec} \mu}{(A^2+B^2)^{\frac{1}{2}}}\right)^2 \tag{3.14}$$

$$= \tfrac{1}{2} \sin \mu\,(Ka)^{-2}\,(\text{amplitude ratio})^2. \tag{3.15}$$

We note that our normalization of $C_v - iC_a$ is perhaps not the most convenient one since it can be shown that in the case of beam seas $C_a \to \frac{1}{4}\pi$ as $Ka \to \infty$, whereas the conventional added-mass coefficient tends to unity. The force on the cylinder is defined without ambiguity by (3.6) and (3.10).

4. Asymptotic behaviour at short wavelengths

It was found previously, in the calculations for beam seas ($\mu = \frac{1}{2}\pi$), that the methods of computation described above are not convenient for large values of Ka. For such values it was, however, possible to find asymptotic approximations (analytically in Ursell (1953), by plausible reasoning in Ursell (1954)) which could be joined up smoothly with the numerical solution for moderate Ka to give the added mass and damping over the whole frequency range (Ursell 1957). Similar difficulties arise in the present problem when Ka is large, but the asymptotic treatment is now more difficult since the second parameter $ka = Ka \cos \mu$ involved here need not also be large near beam incidence. In the present work we shall confine ourselves to the case when both Ka and ka are large. To find the force components, let us consider the potential

$$\hat{\phi}(x,y) = -\frac{b\omega}{Ka\pi}(A+iB)\frac{K_1(kr)}{kK_1'(ka)}\cos\theta, \tag{4.1}$$

which satisfies

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)\hat{\phi} = 0 \quad &\text{when} \quad r > a,\, y > 0; \\ \hat{\phi} = 0 \quad &\text{when} \quad y = 0,\, |x| > a; \end{aligned}\right\} \tag{4.2}$$

$$\frac{\partial \hat{\phi}}{\partial r} = -\frac{b\omega}{Ka\pi}(A+iB)\cos\theta \quad \text{when} \quad r = a,\ -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{4.3}$$

Except in a surface layer, the potential $\hat{\phi}$ may be expected to give a good approximation to ϕ for large Ka since (4.2) is the formal limit of (2.12) for large Ka. We find that

$$\hat{L} \equiv \frac{K\pi}{2b\omega}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\operatorname{Im} \hat{\phi}) \cos\theta\, d\theta = 0,$$

$$\hat{M} \equiv \frac{K\pi}{2b\omega}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\operatorname{Re}\hat{\phi})\cos\theta\, d\theta = -\tfrac{1}{4}\pi(A+iB)\frac{K_1(ka)}{kaK_1'(ka)},$$

and so, to a first approximation,

$$C_v^{(1)} = 0, \quad C_a^{(1)} = -\frac{\pi}{4ka}\frac{K_1(ka)}{K_1'(ka)} = \frac{\pi}{4ka}\left(1-\frac{1}{2ka}-\frac{1}{8(ka)^2}+\ldots\right). \tag{4.4}$$

The next term, which has been found rigorously by Green (1971), is of order $(ka)^{-4}$.

$$C_v^{(2)} = \frac{2}{(Ka)^4\sin\mu},$$

$$C_a^{(2)} = \frac{2}{(ka)^4}\int_0^\infty \frac{dw}{\cosh^3 w(\cosh w-\cosh\beta)}$$

$$= -\frac{2}{(ka)^4\pi}\left(\frac{\beta}{\cosh^3\beta\sinh\beta}+\frac{\pi}{2\cosh^3\beta}+\frac{1}{\cosh^2\beta}+\frac{\pi}{4\cosh\beta}\right), \tag{4.5}$$

where the integral is to be interpreted as a Cauchy principal value, and smaller terms have been neglected. (These terms can also be found more simply by plausible reasoning.) It is seen that $C_a^{(2)}$ is negligible, whereas $C_v^{(2)}$ is the leading term in the expansion of C_v; see also § 5 below.

5. The wave amplitude at infinity

The leading term in C_v for short wavelengths can also be obtained by considering the waves at infinity. The non-rigorous argument given here is an extension of the argument given in earlier work (Ursell 1954) for $ka = 0$, and is applicable for large Ka and arbitrary ka. We consider the waves travelling towards $x = +\infty$. Since Ka is large the wave amplitude depends only on the behaviour of the potential near $\theta = \frac{1}{2}\pi$. (Cf. Ursell (1954, equation (3.1)), and also equation (5.4) below.) Consider the wave-free potential

$$\phi = -\frac{b\omega}{Ka\pi}\frac{A+iB}{kK_1'(ka)}[K_1(kr)\cos\theta+\tfrac{1}{2}\cos\mu(K_0(kr)+K_2(kr)\cos 2\theta)], \tag{5.1}$$

for which

$$-\frac{Ka\pi}{b\omega}(A+iB)^{-1}\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \cos\theta+\tfrac{1}{2}\cos\mu\left(\frac{K_0'(ka)}{K_1'(ka)}+\frac{K_2'(ka)}{K_1'(ka)}\cos 2\theta\right)$$

$$= \left[\cos\theta+\cos\mu\frac{K_0'}{K_1'}\cos^2\theta\right]$$

$$-\left\{\tfrac{1}{2}\cos\mu\frac{K_2'-K_0'}{K_1'}-\cos\mu\frac{K_2'-K_0'}{K_1'}\cos^2\theta\right\}. \tag{5.2}$$

Since the potential (5.1) is wave-free we conclude that the radial-velocity distribution [...] in (5.2) generates the same waves at infinity as the velocity distribution {...}.

We now examine the individual terms in these distributions. In [...] both terms vanish near $\theta = \frac{1}{2}\pi$, but the second term is much smaller than the first, since $\cos^2\theta \ll \cos\theta$ near $\theta = \frac{1}{2}\pi$ and since $|K_0'/K_1'|$ is uniformly bounded for all ka. Thus the contribution to the waves from the second term of [...] is negligible. Similarly, the second term in {...} is negligible. We conclude that the radial-velocity distribution $-(b\omega/Ka\pi)(A+iB)\cos\theta$ of (5.2) gives rise (to a first approximation) to the same waves as the constant radial-velocity distribution

$$-\frac{b\omega}{2Ka\pi}(A+iB)\cos\mu\frac{K_2'-K_0'}{K_1'}, \tag{5.3}$$

which does not vanish near $\theta = \frac{1}{2}\pi$. The velocity (5.3) is the normal velocity on $r = a$; as in Ursell (1954) we consider instead the waves generated by the same normal velocity distributed over the vertical tangent $x = a$. It can be shown that the potential at infinity due to a horizontal-velocity distribution $U_0(y)$ over $x = a$ is given by the wave-train potential

$$-2i\exp(-Ky+iK(x-a)\sin\mu)\operatorname{cosec}\mu\int_0^\infty U_0(y')\exp(-Ky')\,dy'. \tag{5.4}$$

On substituting (5.3) for $U_0(y)$ and comparing with (2.14) we see that

$$A+iB \sim -i\frac{Ka\pi}{\cos\mu}\left(\frac{K_2'(ka)-K_0'(ka)}{K_1'(ka)}\right)^{-1}\exp(iKa\sin\mu) \tag{5.5}$$

as $Ka \to \infty$. Thus

$$\begin{aligned} C_v &= \frac{1}{2}\frac{\sin\mu}{(Ka)^2}\left(\frac{Ka\pi\operatorname{cosec}\mu}{|A+iB|}\right)^2, \quad \text{cf. (3.14)} \\ &\sim \frac{\cos^2\mu}{2\sin\mu}\frac{1}{(Ka)^2}\left(\frac{K_2'(ka)-K_0'(ka)}{K_1'(ka)}\right)^2 \qquad (5.6) \\ &= \frac{2}{(Ka)^4\sin\mu} \quad \text{when } ka \text{ is large,} \end{aligned}$$

in agreement with (4.5) above. The preceding argument appears to be applicable for all values of ka, when Ka is large. When $ka \to 0$ it is readily verified that (5.6) gives the correct result $8(Ka)^{-4}$ for the known case of beam seas.

6. The vertical force on a fixed cylinder

So far we have been concerned with generalized heaving. We shall now obtain the vertical force on a fixed cylinder by using a reciprocity argument. (Relations obtained in this way are known as *Haskind relations* in ship hydrodynamics.) Equation (3.6) shows that the upward vertical force per unit length on a fixed cylinder is

$$i\rho a\omega\exp(ikz-i\omega t)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\phi_f(a\sin\theta, a\cos\theta)\cos\theta\,d\theta, \tag{6.1}$$

where $\phi_f(x,y)$ is the potential defined in § 2 above. Let Green's theorem

$$\int\left\{\phi_f\frac{\partial\phi}{\partial n}-\phi\frac{\partial\phi_f}{\partial n}\right\}ds = 0 \tag{6.2}$$

be applied to the functions ϕ_f and ϕ defined in § 2, and let the integration be taken along the boundary of the region occupied by the fluid lying between vertical planes $x = +R$ and $x = -R$, where the length R will be made to tend to infinity.

$2a/\lambda$	Ka	Asymptote $\mu = 5°$	$\mu = 5°$	Asymptote $\mu = 15°$	$\mu = 15°$	Asymptote $\mu = 25°$	$\mu = 25°$	Asymptote $\mu = 35°$	$\mu = 35°$	Asymptote $\mu = 45°$	$\mu = 45°$
0·08	0·25	—	−15·2	—	−38·4	—	−51·6	—	−59·0	—	−63·3
0·24	0·75	—	−9·1	—	−23·7	—	−32·1	—	−36·1	—	−37·7
0·4	1·25	—	−9·7	—	−22·4	—	−25·8	—	−24·2	—	−20·7
0·56	1·75	—	−11·3	—	−22·0	—	−19·8	—	−12·5	—	−4·1
0·72	2·25	—	−13·0	—	−20·5	—	−12·4	—	0·32	—	13·4
0·87	2·75	—	−14·5	—	−17·7	—	−3·7	—	14·2	—	31·7
1·03	3·25	—	−15·8	—	−14·0	—	5·7	—	28·7	—	50·5
1·19	3·75	—	−16·7	—	−9·6	—	15·9	—	43·7	—	69·7
1·35	4·25	−68·7	−17·4	−26·9	−4·6	12·9	26·4	49·7	59·0	82·2	89·1
1·51	4·75	−66·2	−17·8	−19·5	0·81	25·1	37·2	66·1	74·5	102·4	108·7
3·18	10·00	−39·9	—	58·4	—	152·0	—	239·0	—	315·2	—
4·77	15·00	−14·8	—	133·0	—	273·0	—	403·0	—	518·0	—
6·36	20·00	10·2	—	207·0	—	394·0	—	567·0	—	720·0	—
7·96	25·00	35·2	—	281·0	—	516·0	—	732·0	—	923·0	—
9·54	30·00	60·3	—	355·0	—	637·0	—	896·0	—	—	—
11·14	35·00	85·3	—	429·0	—	758·0	—	—	—	—	—
12·72	40·00	110·0	—	504·0	—	879·0	—	—	—	—	—

$2a/\lambda$	Ka	Asymptote $\mu = 55°$	$\mu = 55°$	Asymptote $\mu = 65°$	$\mu = 65°$	Asymptote $\mu = 75°$	$\mu = 75°$	Asymptote $\mu = 85°$	$\mu = 85°$	$\mu = 90°$
0·08	0·25	—	−67·0	—	−67·6	—	−68·6	—	−69·1	—
0·16	0·52	—	—	—	—	—	—	—	—	−52
0·24	0·75	—	−38·2	—	−38·4	—	−38·5	—	−38·8	—
0·25	0·78	—	—	—	—	—	—	—	—	−37
0·4	1·25	—	−17·0	—	−13·9	—	−11·9	—	−11·2	—
0·5	1·57	—	—	—	—	—	—	—	—	7
0·56	1·75	—	3·8	—	10·1	—	14·3	—	16·2	—
0·66	2·09	—	—	—	—	—	—	—	—	35
0·72	2·25	—	25·1	—	34·4	—	40·8	—	43·8	—
0·75	2·36	—	—	—	—	—	—	—	—	50
0·87	2·75	—	47·0	—	59·2	—	67·5	—	71·5	—
1·00	3·14	—	—	—	—	—	—	—	—	94
1·03	3·25	—	69·4	—	84·3	—	94·5	—	99·5	—
1·19	3·75	—	92·0	—	109·0	—	122·0	—	128·0	—
1·25	3·93	—	—	—	—	—	—	—	—	138
1·35	4·25	109·0	115·0	131·0	135·0	145·0	149·0	152·0	156·0	—
1·5	4·71	—	—	—	—	—	—	—	—	183
1·51	4·75	133·0	138·0	157·0	160·0	173·0	176·0	181·0	184·0	—
3·18	10·00	379·0	—	429·0	—	463·0	—	481·0	—	—
4·77	15·00	614·0	—	689·0	—	740·0	—	766·0	—	—
6·36	20·00	849·0	—	948·0	—	—	—	—	—	—

TABLE 1. $\mathrm{Tan}^{-1}(B/A)$ (degrees)

The integrand vanishes on the segments $y = 0$, $a < |x| \leqslant R$, on account of (2.6) and (2.12), and thus the only non-vanishing contributions to the integral in (6.2) come from the semicircle and from the planes $x = -R$ and $x = +R$. The contribution from the semicircle is

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left(\phi_f \frac{\partial \phi}{\partial r} - \phi \frac{\partial \phi_f}{\partial r}\right) a\, d\theta = -\frac{\omega b}{Ka\pi}(A + iB) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \phi_f(a \sin\theta, a\cos\theta)\, a\cos\theta\, d\theta$$

from (2.7) and (2.13).

The contribution from $x = R$ is seen (from (2.9) and (2.14)) to vanish as $R \to \infty$; the contribution from $x = -R$ is

$$\int_0^\infty \left(\phi_f \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi_f}{\partial x}\right)_{x=-R} dy,$$

$2a/\lambda$	Ka	Asymptote $\mu = 5°$	$\mu = 5°$	Asymptote $\mu = 15°$	$\mu = 15°$	Asymptote $\mu = 25°$	$\mu = 25°$	Asymptote $\mu = 35°$	$\mu = 35°$	Asymptote $\mu = 45°$	$\mu = 45°$
0·08	0·25	—	7·72	—	3·25	—	2·58	—	2·37	—	2·28
0·24	0·75	—	16·6	—	6·44	—	4·73	—	4·11	—	3·8
0·4	1·25	—	22·5	—	9·34	—	7·23	—	6·41	—	5·91
0·56	1·75	—	27·8	—	12·7	—	10·4	—	9·45	—	8·76
0·72	2·25	—	33·4	—	16·8	—	14·4	—	13·3	—	12·3
0·87	2·75	—	39·5	—	21·7	—	19·2	—	17·9	—	16·7
1·03	3·25	—	46·2	—	27·4	—	24·7	—	23·2	—	21·8
1·19	3·75	—	53·5	—	33·8	—	31·0	—	29·3	—	27·6
1·35	4·25	23·5	61·5	23·4	41·1	23·1	38·2	22·8	36·3	22·2	34·2
1·51	4·75	29·8	70·2	29·7	49·1	29·4	46·1	29·0	43·9	28·3	41·6
1·91	6·0	49·0	—	48·8	—	48·5	—	47·8	—	46·8	—
2·23	7·0	67·9	—	67·7	—	67·2	—	66·4	—	65·1	—
2·55	8·0	90·0	—	89·7	—	89·1	—	88·1	—	86·6	—
2·86	9·0	115·0	—	115·0	—	114·0	—	113·0	—	111·0	—
3·18	10·0	143·0	—	143·0	—	142·0	—	141·0	—	139·0	—
4·77	15·0	332·0	—	331·0	—	330·0	—	328·0	—	324·0	—
6·36	20·0	599·0	—	598·0	—	596·0	—	593·0	—	588·0	—
7·96	25·0	944·0	—	943·0	—	941·0	—	937·0	—	931·0	—

$2a/\lambda$	Ka	Asymptote $\mu = 55°$	$\mu = 55°$	Asymptote $\mu = 65°$	$\mu = 65°$	Asymptote $\mu = 75°$	$\mu = 75°$	Asymptote $\mu = 85°$	$\mu = 85°$	$\mu = 90°$
0·08	0·25	—	2·25	—	2·24	—	2·24	—	2·25	—
0·16	0·52	—	—	—	—	—	—	—	—	2·83
0·24	0·75	—	3·6	—	3·48	—	3·39	—	3·36	—
0·25	0·78	—	—	—	—	—	—	—	—	3·44
0·4	1·25	—	5·52	—	5·18	—	4·9	—	4·71	—
0·5	1·57	—	—	—	—	—	—	—	—	5·68
0·56	1·75	—	8·11	—	7·48	—	6·87	—	6·39	—
0·66	2·09	—	—	—	—	—	—	—	—	7·57
0·72	2·25	—	11·4	—	10·4	—	9·35	—	8·43	—
0·75	2·36	—	—	—	—	—	—	—	—	8·64
0·87	2·75	—	15·4	—	14·0	—	12·4	—	10·8	—
1·00	3·14	—	—	—	—	—	—	—	—	12·4
1·03	3·25	—	20·1	—	18·2	—	16·0	—	13·7	—
1·19	3·75	—	25·6	—	23·1	—	20·2	—	16·9	—
1·25	3·93	—	—	—	—	—	—	—	—	16·9
1·35	4·25	21·4	31·7	20·1	28·7	18·2	25·0	15·4	20·6	—
1·5	4·71	—	—	—	—	—	—	—	—	22·0
1·51	4·75	27·2	38·7	25·6	35·1	23·2	30·5	19·4	24·8	—
1·91	6·0	45·3	—	42·9	—	38·9	—	31·9	—	—
2·23	7·0	63·2	—	60·1	—	54·6	—	44·4	—	—
2·55	8·0	84·2	—	80·3	—	73·3	—	59·3	—	—
2·86	9·0	108·0	—	103·0	—	95·0	—	76·6	—	—
3·18	10·0	135·0	—	130.0	—	120·0	—	96·3	—	—
4·77	15·0	318·0	—	309·0	—	289·0	—	235·0	—	—
6·36	20·0	580·0	—	565·0	—	535·0	—	443·0	—	—
7·96	25·0	920·0	—	900·0	—	859·0	—	723·0	—	—

TABLE 2. $(A^2+B^2)^{\frac{1}{2}}$

which (from (2.8) and (2.14)) is seen to tend to

$$\int_0^\infty \left\{\frac{gb_1}{\omega}\exp(-Ky+iKx\sin\mu)\frac{\partial}{\partial x}\left(\frac{gb}{\omega}\operatorname{cosec}\mu\exp(-Ky-iKx\sin\mu)\right)\right.$$

$$\left.-\frac{gb}{\omega}\operatorname{cosec}\mu\exp(-Ky-iKx\sin\mu)\frac{\partial}{\partial x}\left(\frac{gb_1}{\omega}\exp(-Ky+iKx\sin\mu)\right)\right\}dy$$

$$= -\frac{2g^2bb_1iK}{\omega^2}\int_0^\infty \exp(-2Ky)\,dy$$

$$= -g^2bb_1i/\omega^2.$$

$2a/\lambda$	Ka	Asymptote $\mu = 5°$	$\mu = 5°$	Asymptote $\mu = 15°$	$\mu = 15°$	Asymptote $\mu = 25°$	$\mu = 25°$	Asymptote $\mu = 35°$	$\mu = 35°$	Asymptote $\mu = 45°$	$\mu = 45°$
0·08	0·25	—	1·16	—	0·93	—	0·72	—	0·58	—	0·48
0·24	0·75	—	1·62	—	1·41	—	1·18	—	1·0	—	0·88
0·4	1·25	—	2·0	—	1·62	—	1·28	—	1·07	—	0·94
0·56	1·75	—	2·26	—	1·67	—	1·25	—	1·01	—	0·89
0·72	2·25	—	2·42	—	1·62	—	1·16	—	0·93	—	0·81
0·87	2·75	—	2·5	—	1·54	—	1·07	—	0·84	—	0·73
1·03	3·25	—	2·53	—	1·44	—	0·98	—	0·77	—	0·66
1·19	3·75	—	2·52	—	1·34	—	0·9	—	0·7	0·99	0·6
1·35	4·25	—	2·48	—	1·25	—	0·83	—	0·64	0·85	0·55
1·51	4·75	—	2·43	1·9	1·17	1·2	0·77	0·9	0·59	0·75	0·51
1·91	6·0	—	—	1·49	—	0·92	—	0·69	—	0·57	—
2·23	7·0	—	—	1·25	—	0·77	—	0·58	—	0·48	—
2·55	8·0	—	—	1·08	—	0·67	—	0·5	—	0·41	—
2·86	9·0	—	—	0·95	—	0·59	—	0·44	—	0·36	—
3·18	10·0	—	—	0·85	—	0·52	—	0·39	—	0·32	—
4·77	15·0	1·62	—	0·55	—	0·34	—	0·25	—	0·2	—
6·36	20·0	1·2	—	0·4	—	0·25	—	0·18	—	0·15	—
7·96	25·0	0·95	—	0·32	—	0·2	—	0·15	—	0·12	—
9·54	30·0	0·79	—	0·27	—	0·16	—	0·12	—	0·099	—
11·14	35·0	0·67	—	0·23	—	0·14	—	0·1	—	0·084	—
12·72	40·0	0·59	—	0·2	—	0·12	—	0·09	—	0·073	—

$2a/\lambda$	Ka	Asymptote $\mu = 55°$	$\mu = 55°$	Asymptote $\mu = 65°$	$\mu = 65°$	Asymptote $\mu = 75°$	$\mu = 75°$	Asymptote $\mu = 85°$	$\mu = 85°$	$\mu = 90°$
0·08	0·25	—	0·43	—	0·39	—	0·36	—	0·35	—
0·16	0·52	—	—	—	—	—	—	—	—	0·58
0·24	0·75	—	0·8	—	0·75	—	0·72	—	0·7	—
0·25	0·78	—	—	—	—	—	—	—	—	0·72
0·4	1·25	—	0·87	—	0·84	—	0·83	—	0·84	—
0·5	1·57	—	—	—	—	—	—	—	—	0·87
0·56	1·75	—	0·83	—	0·81	—	0·83	—	0·86	—
0·66	2·09	—	—	—	—	—	—	—	—	0·87
0·72	2·25	—	0·76	—	0·75	—	0·78	—	0·84	—
0·75	2·36	—	—	—	—	—	—	—	—	0·86
0·87	2·75	—	0·68	—	0·68	—	0·72	—	0·8	—
1·0	3·14	—	—	—	—	—	—	—	—	0·8
1·03	3·25	—	0·62	—	0·62	—	0·66	—	0·75	—
1·19	3·75	0·89	0·56	0·85	0·56	0·88	0·6	1·00	0·7	—
1·25	3·93	—	—	—	—	—	—	—	—	0·73
1·35	4·25	0·76	0·51	0·73	0·51	0·76	0·55	0·87	0·65	—
1·5	4·71	—	—	—	—	—	—	—	—	0·67
1·51	4·75	0·67	0·47	0·64	0·47	0·66	0·51	0·77	0·6	—
1·91	6·0	0·51	—	0·48	—	0·5	—	0·59	—	—
2·23	7·0	0·42	—	0·4	—	0·42	—	0·5	—	—
2·55	8·0	0·36	—	0·34	—	0·35	—	0·42	—	—
2·86	9·0	0·32	—	0·3	—	0·31	—	0·37	—	—
3·18	10·0	0·28	—	0·27	—	0·27	—	0·33	—	—
4·77	15·0	0·18	—	0·17	—	0·17	—	0·2	—	—
6·36	20·0	0·13	—	0·12	—	0·12	—	0·14	—	—
7·96	25·0	0·1	—	0·096	—	0·095	—	0·11	—	—
9·54	30·0	0·086	—	0·079	—	0·077	—	0·088	—	—
11·14	35·0	0·073	—	0·067	—	0·065	—	0·073	—	—
12·72	40·0	0·064	—	0·058	—	0·056	—	0·062	—	—

TABLE 3. Amplitude ratio $\pi Ka \operatorname{cosec} \mu/(A^2+B^2)^{\frac{1}{2}}$

$2a/\lambda$	Ka	Asymptote $\mu = 5°$	$\mu = 5°$	Asymptote $\mu = 15°$	$\mu = 15°$	Asymptote $\mu = 25°$	$\mu = 25°$	Asymptote $\mu = 35°$	$\mu = 35°$	Asymptote $\mu = 45°$	$\mu = 45°$
0·08	0·25	—	0·95	—	1·8	—	1·75	—	1·53	—	1·33
0·24	0·75	—	0·2	—	0·46	—	0·52	—	0·51	—	0·48
0·4	1·25	—	0·11	—	0·22	—	0·22	—	0·21	—	0·2
0·56	1·75	—	0·73	—	0·12	—	0·11	—	0·096	—	0·091
0·72	2·25	—	0·051	—	0·067	—	0·056	—	0·049	—	0·046
0·87	2·75	—	0·036	—	0·04	—	0·032	—	0·027	—	0·025
1·03	3·25	—	0·026	—	0·025	—	0·019	—	0·016	—	0·015
1·19	3·75	—	0·02	—	0·017	—	0·012	—	0·01	—	0·0092
1·35	4·25	—	0·015	—	0·011	—	0·008	—	0·0065	—	0·0059
1·51	4·75	—	0·011	—	0·0079	—	0·0055	—	0·0044	—	0·004
3·18	10·0	—	—	0·00078	—	0·00048	—	0·00034	—	0·00028	—
4·77	15·0	0·00046	—	0·000152	—	0·000094	—	0·000068	—	0·000056	—
6·36	20·0	0·00022	—	0·000048	—	0·000030	—	0·000022	—	0·000076	—
7·96	25·0	0·000058	—	0·0000198	—	0·0000122	—	0·0000090	—	0·0000072	—
9·54	30·0	0·000028	—	0·0000094	—	0·0000058	—	0·0000042	—	0·0000034	—
11·14	35·0	0·0000154	—	0·0000052	—	0·0000032	—	0·0000024	—	0·0000018	—

$2a/\lambda$	Ka	Asymptote $\mu = 55°$	$\mu = 55°$	Asymptote $\mu = 65°$	$\mu = 65°$	Asymptote $\mu = 75°$	$\mu = 75°$	$\mu = 85°$
0·08	0·25	—	1·19	—	1·08	—	1·02	0·98
0·24	0·75	—	0·46	—	0·45	—	0·44	0·44
0·4	1·25	—	0·2	—	0·2	—	0·21	0·22
0·56	1·75	—	0·091	—	0·097	—	0·11	0·12
0·72	2·25	—	0·046	—	0·05	—	0·058	0·07
0·87	2·75	—	0·025	—	0·028	—	0·033	0·042
1·03	3·25	—	0·015	—	0·016	0·01856	0·02	0·026
1·19	3·75	—	0·0092	—	0·01	0·01046	0·013	0·017
1·35	4·25	—	0·006	—	0·0065	0·00634	0·0081	0·012
1·51	4·75	—	0·004	—	0·0044	0·00406	0·0055	0·0081
3·18	10·0	0·00024	—	0·00022	—	0·00020	—	—
4·77	15·0	0·000048	—	0·000044	—	0·000040	—	—
6·36	20·0	0·0000152	—	0·0000138	—	0·000013	—	—
7·96	25·0	0·0000062	—	0·0000056	—	0·0000052	—	—
9·54	30·0	0·0000030	—	0·0000028	—	0·0000026	—	—
11·14	35·0	0·0000016	—	0·0000014	—	0·0000014	—	—

TABLE 4. Force component in phase with velocity, $C_v = (AL - BM)/(A^2 + B^2)$

It follows that
$$-\frac{\omega b}{K\pi}(A+iB)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\phi_f \cos\theta\, d\theta - \frac{ig^2 b b_1}{\omega^2} = 0,$$
whence
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\phi_f(a\sin\theta, a\cos\theta)\cos\theta\, d\theta = -\frac{g b_1 \pi i}{\omega(A+iB)},$$
where b_1 is the amplitude of the incident wave. The force can now be found from (6.1); it is
$$\frac{\rho g a b_1 \pi}{A+iB}\exp(ikz - i\omega t) \tag{6.3}$$
per unit length.

7. Results

The mathematical procedure described in § 3 was programmed in IBM 1130 Fortran to give the motion parameters $\tan^{-1} B/A$ and $(A^2+B^2)^{\frac{1}{2}}$ defined by equations (2.13) and (2.14); the amplitude ratio $\pi Ka \operatorname{cosec}\mu/(A^2+B^2)^{\frac{1}{2}}$ defined by (3.12); and the added-mass coefficient C_a and the damping coefficient C_v defined by (3.11). Tables 1–5 tabulate the computed results for $\mu = 5°, 15°, \ldots, 85°$

$2a/\lambda$	Ka	Asymptote $\mu = 5°$	$\mu = 5°$	Asymptote $\mu = 15°$	$\mu = 15°$	Asymptote $\mu = 25°$	$\mu = 25°$	Asymptote $\mu = 35°$	$\mu = 35°$	Asymptote $\mu = 45°$	$\mu = 45°$
0·08	0·25	—	3·72	—	2·5	—	1·62	—	1·16	—	0·93
0·24	0·75	—	1·26	—	1·02	—	0·8	—	0·66	—	0·58
0·4	1·25	—	0·72	—	0·59	—	0·49	—	0·45	—	0·44
0·56	1·75	—	0·49	—	0·4	—	0·36	—	0·36	—	0·37
0·72	2·25	—	0·37	—	0·31	—	0·29	—	0·3	—	0·33
0·87	2·75	—	0·29	—	0·25	—	0·25	—	0·26	—	0·29
1·03	3·25	—	0·24	—	0·22	—	0·22	—	0·23	—	0·26
1·19	3·75	—	0·2	—	0·19	—	0·19	—	0·21	—	0·23
1·35	4·25	—	0·18	—	0·17	—	0·17	—	0·19	—	0·21
1·51	4·75	0·15	0·16	0·15	0·15	0·16	0·16	0·17	0·17	0·19	0·19
3·18	10·0	0·075	—	0·077	—	0·082	—	0·090	—	0·10	—
4·77	15·0	0·051	—	0·052	—	0·056	—	0·061	—	0·070	—
6·36	20·0	0·038	—	0·039	—	0·042	—	0·046	—	0·053	—
7·96	25·0	0·031	—	0·032	—	0·034	—	0·037	—	0·043	—
9·54	30·0	0·026	—	0·027	—	0·028	—	0·031	—	0·036	—
11·14	35·0	0·022	—	0·023	—	0·024	—	0·027	—	0·031	—
12·72	40·0	0·019	—	0·02	—	0·021	—	0·024	—	0·027	—

$2a/\lambda$	Ka	Asymptote $\mu = 55°$	$\mu = 55°$	Asymptote $\mu = 65°$	$\mu = 65°$	Asymptote $\mu = 75°$	$\mu = 75°$	$\mu = 85°$	$\mu = 90°$
0·08	0·25	—	0·8	—	0·73	—	0·7	0·69	—
0·16	0·52	—	—	—	—	—	—	—	0·5
0·24	0·75	—	0·53	—	0·5	—	0·48	0·47	—
0·25	0·78	—	—	—	—	—	—	—	0·46
0·4	1·25	—	0·45	—	0·46	—	0·48	0·49	—
0·5	1·57	—	—	—	—	—	—	—	0·53
0·56	1·75	—	0·4	—	0·44	—	0·49	0·54	—
0·66	2·09	—	—	—	—	—	—	—	0·58
0·72	2·25	—	0·36	—	0·41	—	0·49	0·57	—
0·75	2·36	—	—	—	—	—	—	—	0·6
0·87	2·75	—	0·33	—	0·39	—	0·47	0·59	—
1·0	3·14	—	—	—	—	—	—	—	0·64
1·03	3·25	—	0·3	—	0·36	—	0·45	0·6	—
1·19	3·75	—	0·27	—	0·33	—	0·43	0·6	—
1·25	3·93	—	—	—	—	—	—	—	0·67
1·35	4·25	—	0·25	—	0·31	—	0·41	0·6	—
1·5	4·71	—	—	—	—	—	—	—	0·69
1·51	4·75	0·23	0·23	0·28	0·29	0·33	0·39	0·59	—
3·18	10·0	0·12	—	0·16	—	0·24	—	—	—
4·77	15·0	0·086	—	0·11	—	0·17	—	—	—
6·36	20·0	0·065	—	0·087	—	0·14	—	—	—
7·96	25·0	0·053	—	0·071	—	0·11	—	—	—
9·54	30·0	0·044	—	0·059	—	0·094	—	—	—
11·14	35·0	0·038	—	0·051	—	0·082	—	—	—
12·72	40·0	0·033	—	0·045	—	0·072	—	—	—

TABLE 5. Force component in phase with acceleration, $C_a = (AM+BL)/(A^2+B^2)$

and $Ka = 0{\cdot}25, 0{\cdot}75, \ldots, 4{\cdot}75$, together with values for the case of ordinary heaving ($\mu = 90°$) taken from Ursell (1949, 1957). The asymptotic formulae for the same five quantities (given in §§ 4 and 5) were also computed for the same values of μ and for $Ka > 5$, and are tabulated in the same tables. As was to be expected, there are discrepancies between the computed values and the asymptotic formulae near $Ka = 5$. It is suggested that parameter values for $Ka > 5$ are best found by a method of graphical interpolation; cf. § 4 above. For the case $\mu = 90°$ the phase angle $\tan^{-1} B/A$, the amplitude ratio and the added-mass coefficient were interpolated in this way in Ursell (1957) and the same three parameters are found to be the most suitable for interpolation in the present work. (It was found, however, that for $\mu = 5°$ the phase angle could not easily

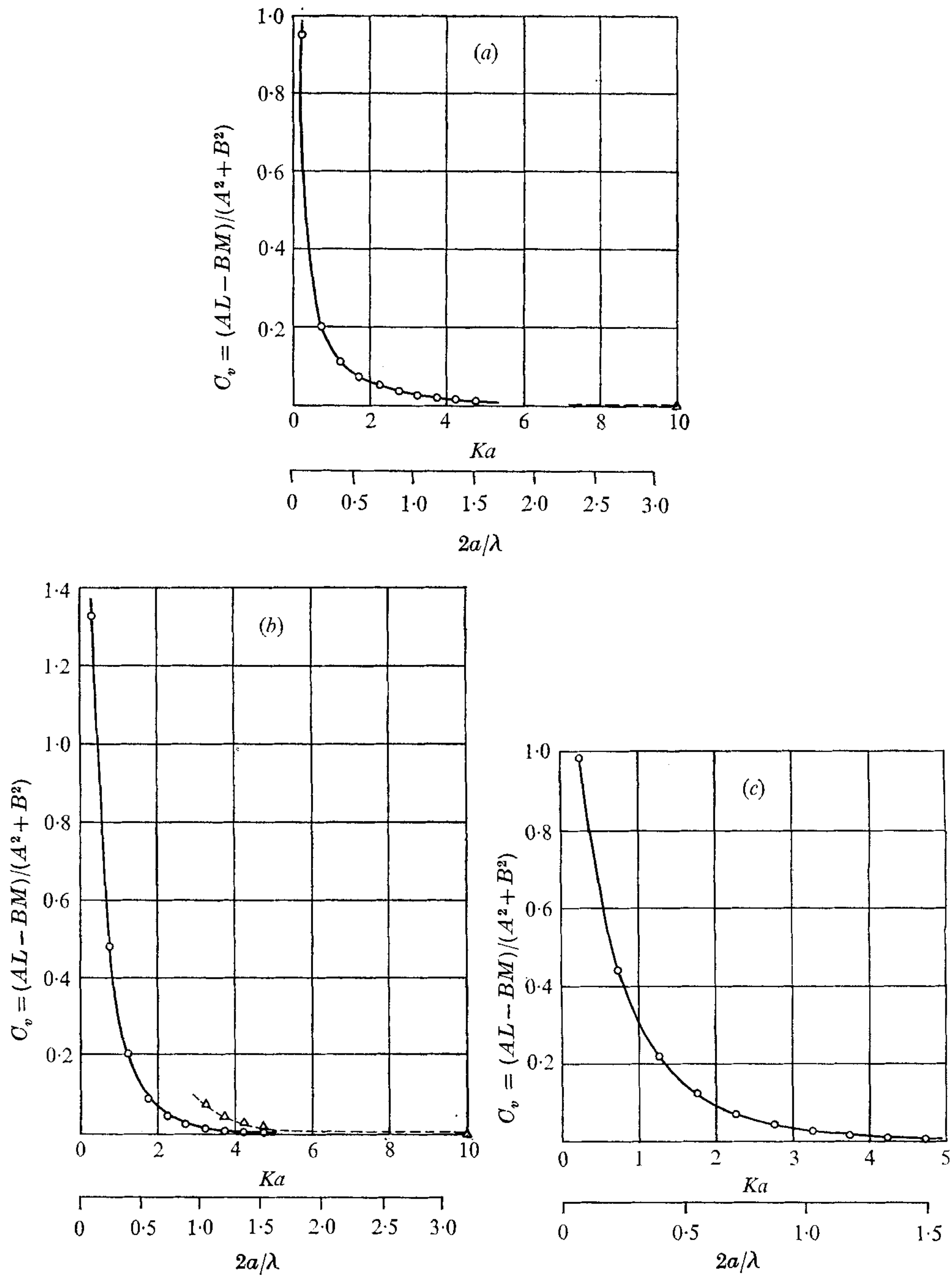

FIGURE 1. Damping coefficient. —○—, computed values; ‐‐△‐‐, asymptotic values. (*a*) $\mu = 5°$. (*b*) $\mu = 45°$. (*c*) $\mu = 85°$.

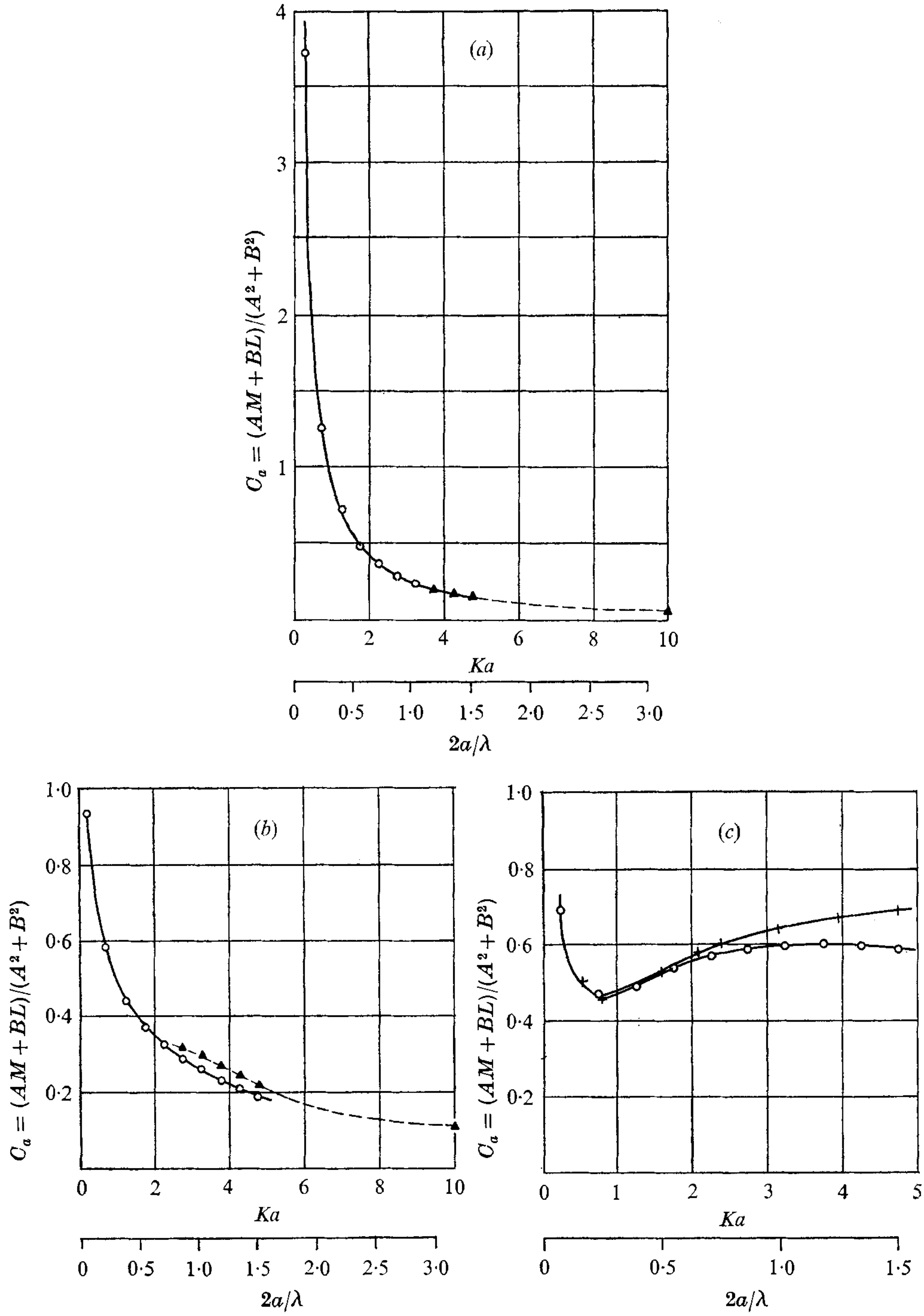

FIGURE 2. Added-mass coefficient. —○—, computed values. --▲--, asymptotic values. (*a*) $\mu = 5°$. (*b*) $\mu = 45°$. (*c*) —○—, $\mu = 85°$; —+—, $\mu = 90°$, reference values (Ursell 1949).

be interpolated between $Ka = 4{\cdot}75$ and $Ka = \infty$, although the discussions in § 5 would be expected to be valid for small values of μ.)

For convenience the values of the generalized damping coefficients for $\mu = 5°$, 45° and 85° are also presented graphically in figures 1 (*a*)–(*c*), and the values of the generalized added-mass coefficient for the same angles in figures 2 (*a*)–(*c*). These parameters refer to the generalized heaving problem. The reader is reminded that the vertical force due to oblique waves on a fixed cylinder is given by (6.3) above, and involves only the parameters A and B.

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